Lecture 16

- The tools of the previous 3 lectures have built us up to approaching a claim made by Fourier in 1807: it should be possible to represent **all** solutions to the 1D heat eqn. by trigonometric series

Series Solution to the Heat Eqn

- Consider (H) $\partial u/\partial t - \Delta u = 0$ in $[0,\infty)_t \times U$ $\subseteq \mathbb{R}^n$
  for U a domain, with Dirichlet or Neumann B.C.

- By separation of variables (Lemma 5.1), product solutions $u(t,x) = v(t)\,\Phi(x)$ have
$$-\Delta\Phi = \lambda\Phi$$
$$\partial v/\partial t = -\lambda v$$
giving the family of solutions $u_n(t,x) = e^{-\lambda_n t}\Phi_n(x)$
(and in 1D, $\Phi_n(x)$ often looked trigonometric)

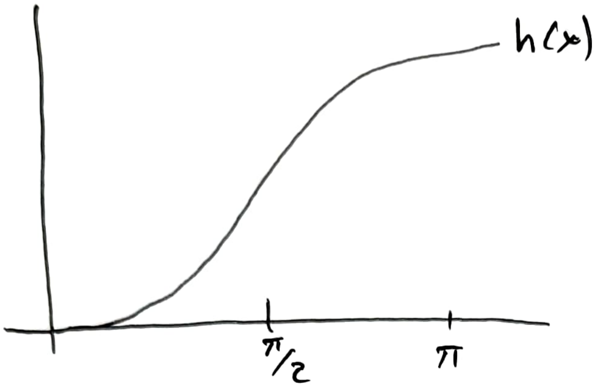
- Fourier's strategy entreats us to express
$$u(t,x) = \sum_{n=1}^{\infty} a_n e^{-\lambda_n t}\Phi_n(x)$$
for any solution u given some choice of a_n.

- Let us assume $u(0,x) = h(x)$. Then
$$h(x) = \sum_{n=1}^{\infty} a_n \Phi_n(x) \qquad (A)$$
If we can show $\{\Phi_n\}$ is an orthonormal basis to $L^2(U)$, ~~the~~ lecture 14 gives us a way to find a_n.
So that this convergence (A) holds.

$\rightsquigarrow$ this doesn't resolve all issues. For example, it doesn't guarantee $u(t,x)$ solves (H). We approach these in the trigonometric context.

ex.) Consider a 1D metal rod w/ insulated ends of length π
$$\begin{cases} \partial u/\partial t - \partial^2 u/\partial x^2 = 0 \\ \partial u/\partial x(t,0) = \partial u/\partial x(t,\pi) = 0 \end{cases}$$

•) Let us take some initial heat distribution
$h(x) = 3\pi x^2 - 2x^3 = u(0,x)$



•) The b.c. give $\Phi_n(x) = \cos(nx)$ for $n \in \mathbb{N}_0$
We claim these are an orthogonal set in $L^2([0,\pi])$:

$$\int_0^\pi \cos(nx)\cos(mx)dx = ?$$

if $m=n=0$, this is $\int_0^\pi dx = \pi$

if $m=n\neq 0$, $n\geq 1$

$$\int_0^\pi \cos^2(nx)dx = \int_0^\pi \cancel{\frac{\cos(nx/2)+1}{2}dx} =$$
$$= \int_0^\pi \frac{\cos(2nx)+1}{2}dx = \pi/2$$

if $m\neq n$

$$\int_0^\pi \frac{\cos((m+n)x)+\cos((m-n)x)}{2}dx = \frac{1}{2}\left[\frac{\sin((m+n)x)}{m+n} + \frac{\sin((m-n)x)}{m-n}\right]_0^\pi$$
$$= 0$$

Showing orthogonality.

•) Then, to get $h(x) = \sum_{n=0}^\infty a_n \Phi_n(x)$,

$$a_0 = \frac{\langle h, \Phi_0\rangle}{\|\Phi_0\|^2} = \frac{1}{\pi}\int_0^\pi h(x)dx$$
$$a_n = \frac{2}{\pi}\int_0^\pi h(x)\cos(nx)dx \qquad n\geq 1$$

or
$$a_n = \begin{cases} \pi^3/2 & n=0 \\ -48/\pi n^4 & n\geq 1 \text{ odd} \\ 0 & n\geq 2 \text{ even}\end{cases}$$

∘) If we shifted to $u(t,x)$, we would have

$$u(t,x) = \pi^3/2 - \sum_{\substack{n \in \mathbb{N} \\ n \text{ odd}}} \frac{48}{\pi n^4} e^{-n^2 t} \cos(nx)$$

which converges readily.

Periodic Fourier Series

- Since we intend to focus on Fourier Series based on Sines + Cosines, we consider periodic functions on $\mathbb{R}$.
  Thus, we force a "periodic domain"
  $$\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z} \quad \text{(the torus in 1D)}$$
  meaning: Any two points y, $y+2\pi k$ for $k \in \mathbb{Z}$ are "the same" in $\mathbb{T}$.

- The space $C^m(\mathbb{T})$ is the space of functions in $C^m(\mathbb{R})$ that are 2π-periodic.

- We consider $L^2(\mathbb{T})$ and $\langle f, g \rangle = \int_{-\pi}^{\pi} f\bar{g}\,dx$

- The Helmholtz equation on $\mathbb{T}$ is $-\partial^2\phi/\partial x^2 = \lambda\phi$ with periodicity giving B.C. to ϕ. The solutions are
  $$\phi_k(x) = e^{ikx} \text{ for } k \in \mathbb{Z} \quad (\lambda_k = k^2).$$

Notice $\langle \phi_k, \phi_\ell \rangle = \int_{-\pi}^{\pi} e^{i(k-\ell)x}\,dx = \begin{cases} 2\pi & k=\ell \\ 0 & k \neq \ell \end{cases}$

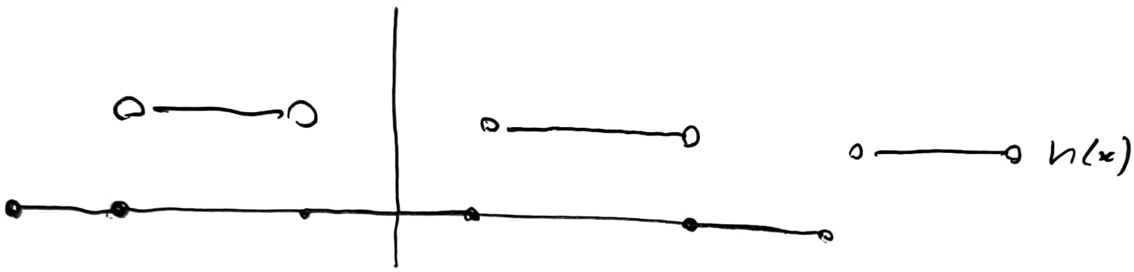
We define Fourier Coefficients $c_k[f] := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x) e^{-ikx}\,dx$

- Note that we index by $\mathbb{Z}$, not $\mathbb{N}$, so
  $$S_n[f] = \sum_{k=-n}^{n} c_k[f] e^{ikx} \quad \text{(2-sided sum)}$$

∘) Bessel's Inequality becomes $\sum_{k \in \mathbb{Z}} |c_k[f]|^2 \leq \frac{1}{2\pi} \|f\|_2^2$ with equality iff $S_n[f] \to f$ in $L^2(\mathbb{T})$.

ex.) Potential Complications with Convergence

Define $h(x) = \begin{cases} 0 & x \in [-\pi/2, \pi/2] + 2\pi\mathbb{Z} \\ 1 & x \in (\pi/2, 3\pi/2] + 2\pi\mathbb{Z} \end{cases}$



Then, $c_k[h] = \frac{1}{2\pi} \int_{-\pi}^{\pi} h(x) e^{-ikx} dx = \frac{1}{2\pi} \int_0^{2\pi} h(x) e^{-ikx} dx$

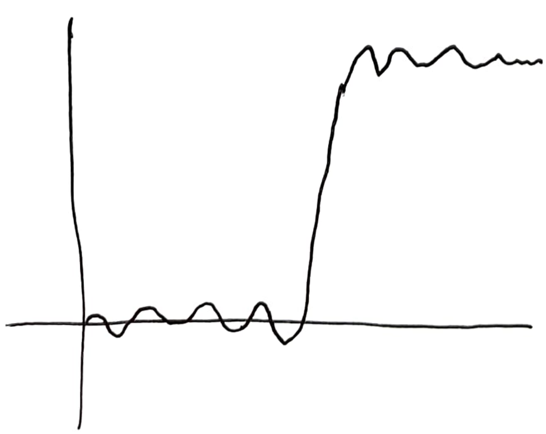
$$= \frac{1}{2\pi} \int_{\pi/2}^{3\pi/2} e^{-ikx} dx = \begin{cases} 1/2 & k = 0 \\ \frac{(-1)^k}{\pi k} \sin(\pi k/2) & k \neq 0 \end{cases}$$

Notice $c_{-k}[h] = c_k[h]$ and

$$S_n[h](x) = \frac{1}{2} + 2\sum_{k=1}^{n} \frac{(-1)^k}{\pi k} \sin\left(\frac{\pi k}{2}\right) \cos(kx)$$

$$\cos(kx) = \frac{e^{ikx} + e^{-ikx}}{2}$$

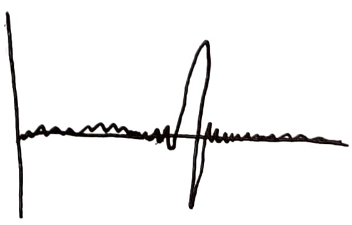
$S_{20}[h](x)$



~~Plot~~ Plotting $h(x) - S_n[h](x)$

$n = 20$

$n = 100$



•) We will have $\lim_{n\to\infty} \int_{-\pi}^{\pi} |h(x) - S_n[h](x)|^2 dx \to 0$,
as we will show later $\square$

•) Another Convergence we are interested in is pointwise Convergence. It is said that $\{f_n\}$ converges to f pointwise if for every x in the domain

$$\lim_{n\to\infty} f_n(x) = f(x)$$

$\rightsquigarrow$ In the above ex, $S_n[h] \to h$ pointwise for all x except $x = \pi/2$, caused by the Jump discontinuity.

we will consider this and another type of convergence in the next lecture.